The Limiting Form of the Eddy Diffusivity Close to a Wall

R. H. NOTTER and C. A. SLEICHER

University of Washington, Seattle, Washington

A controversy has existed for decades over the question of whether the eddy diffusivity and eddy viscosity near a wall vary as the third or fourth power of the distance from the wall. In this paper we present a theoretical analysis which suggests that for high Prandtl (or Schmidt) number fluids and for the boundary conditions of uniform wall temperature (or concentration) or heat flux, the correct law for the eddy diffusivity depends on the third power of the distance from the wall.

The eddy viscosity and eddy diffusivity may be defined respectively by

$$\epsilon = \frac{-\overline{uv}}{d\overline{U}/dy} \qquad \epsilon_H = \frac{-\overline{v\theta}}{d\overline{T}/dy} \tag{1}$$

Equations (1) are written with the velocity and temperature given in the standard form of a mean part plus a fluctuating part. All the variables are listed under the Notation. The long bar over variables denotes an average over any variable on which the turbulence is homogeneous (x, z, or t). An eddy diffusivity for the transport of any other passive scalar quantity can be defined in an analogous manner. Objections may be raised to these definitions, and a good discussion of the matter is given by Hinze (*3*). Nevertheless, the concept of eddy diffusivity has been of great practical worth in calculating heat and mass transfer rates when estimates of the eddy viscosity can be made from the velocity field. The usual practice is to assume that ϵ is equal or nearly equal to ϵ_H.

In 1951 Reichardt (*9*) showed by expanding u and v in a Taylor series about $y = 0$ that

$$\overline{uv} = C_1 y^3 + C_2 y^4 + \ldots . \tag{2}$$

but the magnitude of C_1 and C_2 were unspecified, and there has been speculation as to whether or not C_1 might be zero. For example, Ohji (*8*) has argued on theoretical grounds that for wall turbulence that is homogeneous in the x and z directions, the constant C_1 drops out in the averaging procedure and thus $\overline{uv}$ varies as y^4. On the other hand, Wasan, Tien, and Wilke (*14*) found C_1 not to be zero. They attempted to estimate C_1 and C_2 by matching U and $d\overline{U}/dy$ calculated from Equation (2) to values calculated from a widely accepted form of the logarithmic law (which is empirical) for U (*1*). The point at which a match is obtained, however, is well outside of the region of validity of both Equation (2) and the logarithmic law, and the results are, therefore, open to question.

The behavior of ϵ near a wall can also be determined experimentally. In the region $8 < y^+ < 40$ calculations of ϵ from velocity profiles indicate that ϵ is roughly *quadratic* in y. Closer to a wall than this it is difficult to calculate ϵ accurately from velocity profiles. It can, however, be inferred from measurements of heat or mass flux at the wall. Thus, Son and Hanratty (*10*) recently presented an empirical correlation in which ϵ (taken to equal ϵ_H) varies with the fourth power of y. This work was criticized by Hughmark (*7*) and by Hubbard (*6*) on the grounds that Son and Hanratty had not used the best available data, and they concluded that ϵ_H varies as the cube of y.

The variation of $\overline{uv}$ with y is not necessarily the same as the variation of $\overline{v\theta}$ with y. Nonetheless, there has not been much work aimed at determining $\overline{v\theta}$ directly. An analysis similar to Reichardt's was done for $\overline{v\theta}$ by Tien (*13*) in 1964, but the order of magnitude of the y^3 and y^4 terms in an expression for ϵ_H was not determined. The primary purpose of this work is to obtain a comparison of the y^3 and y^4 coefficients in the eddy diffusivity in order to obtain the limiting form of this function near the wall. For this purpose only an approximate analysis is required, for unless the coefficient of the y^3 term is effectively zero, then in the limit as $y \to 0$ the diffusivity will vary as y^3.

A secondary purpose of this paper is to acquaint readers with an interesting and potentially useful approach to problems of turbulence. An unusual aspect of the analysis is that empiricism is avoided. The method is to develop u, v, w, and θ in Fourier modes in x and z and arbitrary functions of y. The approach was first expounded in two stimulating papers by Sternberg (*11, 12*) which are concerned with velocity distributions in the viscous sublayer, defined as that part of the boundary layer extending from the wall to the fully developed part of the flow. This paper extends his work to the temperature equation in the region close to the wall in order to determine the limiting form of ϵ_H as $y \to 0$.

The basic assumption of Sternberg's analysis is that near the wall the nonlinear terms in the turbulence equations for the fluctuating quantities may be neglected, that is, the term $\partial/\partial x\ (u_i u_j - \overline{u_i u_j})$ is neglected. The linearization procedure is discussed elsewhere (*12*), and the reader is referred there for detailed justification. Some points, however, need emphasis here.

One measure of turbulence is turbulence intensity, and near a wall the lateral intensities $v'/\overline{U}$ and $w'/\overline{U}$ are very small whereas $u'/\overline{U}$ appears to approach a constant of 0.3 (*4*). In other words, turbulence extends to the wall, and there is no region which may be considered as a truly laminar sublayer. There will be some region near the wall, however, where the viscous terms in the turbulence equations will dominate the inertia terms. This is true irrespective of the value of turbulence intensity, which is evident if one reflects on the fact that near an oscillating plate in a semi-infinite fluid without body forces $u'/\overline{U}$ is infinite. Sternberg was interested in developing solutions valid from the wall to the fully-developed part of the flow. He found that his solutions of the linearized equations are questionable primarily at the outer edge of the viscous layer, where y^+ is of the order of 30 to 60. Closer to the wall he was able to obtain better agreement with data. Moreover, he found that very near the wall there is essentially no difference between solutions for the velocity fluctuations found from simplified equations with no inertia terms and solutions from fuller equations in which linearized convective terms are retained. In other words, in the limit as the wall is approached the viscous effects determine the leading terms in an expansion in y for the amplitude of the velocity fluctuations. Similarly, linearization of the temperature equation results in neglecting the term $\partial/\partial x^k\ (\theta u_k - \overline{\theta u_k})$ in the equation for the fluctuating part of the temperature, and arguments can be invoked to show that very close to the wall these terms are small compared to the conduction terms.

In the present work we are able to take advantage of two restrictions which were not available to Sternberg. First, this work is restricted to the region where the variable $Y = \sqrt{\beta/2\nu}\, y$ is much less than unity, and so we are concerned with values of y^+ of the order of 1 or less. This close to the wall the viscous terms greatly exceed the nonlinear terms in the turbulence equations. Second, our analysis for heat transfer is restricted to the case of asymptotically large Prandtl number because it is only for large N_{Pr} (or N_{Sc}) that the form of the eddy diffusivity near the wall is important.

Sternberg assumes that the fluctuating components of velocity and pressure have the form

$$u_k = \text{Real}\,\{f_k(y)e^{i(k_x x + k_z z - \beta t)}\} \tag{3}$$

where the f_k are complex functions of y, and k_k, k_z, and β are real wave numbers and the frequency respectively. If Equation (3) is written in terms of sine and cosine terms, it can be shown to be equivalent to the form

$$u_k = h_k\,(y)\ \text{Real}\,\{e^{i[k_x x + k_z z - \beta t + \zeta(y)]}\} \tag{4}$$

where $h_k\,(y)$ is real and the phase ζ is a function of y. A less general form of Equation (4) may be written by assuming that ζ is linear in y. Similarly, the temperature fluctuations may be represented by the simplified form

$$\theta = \theta_1(y)\ \text{Real}\,\{e^{i[k_x x + k_z z - \beta t + \xi(y)]}\} \tag{5}$$

In this note we will develop solutions for v and θ by using first the form (3) to represent all the fluctuations and then the forms (4) and (5), with the phases ζ and ξ taken as linear functions of y.

The relevant equations can be simplified by several assumptions. The mean temperature and mean velocity are assumed to vary much more rapidly with y than with x, and so they are taken to be functions of y alone. The mean flow is further restricted by the assumption that, since the equations hold only very near the wall, the mean velocity can be taken as being proportional to y. The solution of these equations is facilitated by making them dimensionless with δ_s, the sublayer thickness, $U_w = \beta/k_x$, the velocity of the Fourier component under consideration, and T_o as the characteristic temperature. The fluctuations given by (3) are then made dimensionless and substituted into the linearized equations to derive the equation for the dimensionless amplitude of the normal velocity fluctuations, $\hat{v}(Y)$:

$$\frac{d^3\hat{v}}{dY^3} + 2i\left(1 - \frac{KY}{\lambda}\right)\frac{d\hat{v}}{dY} + \frac{2iK\hat{v}}{\lambda} = -\,2iA_1 \tag{6}$$

The constant A_1 in Equation (6) is given by

$$A_1 = \frac{ik_x p_\phi}{\rho U_w^2 \sqrt{\beta/2\nu}}\quad (1 + j\tan^2\phi) \tag{7}$$

where $j = 0$ for two dimensional fluctuations and $j = 1$ for three dimensional fluctuations. The angle ϕ is between the directions of wave motion and mean motion, and the factor p_ϕ is the amplitude of the pressure fluctuations and is a complex constant close to the wall (*12*).

A general solution for $\hat{v}(Y)$ is obtainable from Equation (6), but since the domain of interest is restricted here to $Y << 1$, it is convenient to assume a solution of the form

$$\hat{v}(Y) = \sum_{n=o}^{\infty} \overline{C}_n\, Y^n \tag{8}$$

The solution has the specific form

$$\hat{v}(Y) = \overline{C}_2\, Y^2 - \frac{iA_1}{3}\, Y^3 + O(Y^4) \tag{9}$$

after continuity and the boundary condition $\hat{v}(Y{=}0) = 0$ are applied. Sternberg (*12*) obtained plots of the magnitude of $d\hat{v}/dY$ and $d^3\hat{v}/dY^3$. From these plots, examined for $Y << 1$, it can be seen that $|\overline{C}_2| \simeq |A_1| \simeq 1$.

The equation for the temperature field is now examined. Subject to the restrictions mentioned previously, the mean temperature obeys

$$\frac{d^2\,\overline{T}/T_o}{dY^2} = 0 \tag{10}$$

The solution for the dimensionless mean temperature is

$$\frac{\overline{T}}{T_o} = \frac{AY}{\lambda} + B \tag{11}$$

so that the mean temperature gradient is uniform (near the wall). The equation for the dimensionless, complex amplitude of the temperature fluctuations, $\hat{\theta}(Y)$ is given by

$$\frac{1}{N_{Pr}}\frac{d^2\hat{\theta}}{dY^2} - \hat{\theta}\left[\frac{2(\tilde{k}_x^2 + \tilde{k}_z^2)}{\tilde{\beta} N_{Re} N_{Pr}} - \frac{2i}{\tilde{\beta}}\left(\frac{K\tilde{k}_x Y}{\lambda} - \tilde{\beta}\right)\right] = \frac{2A\tilde{v}(Y)}{\tilde{\beta}} \tag{12}$$

Equation (12) is, of course, valid for any Prandtl number. However, the only situations where the form of the eddy diffusivity very near the wall is of importance are those of high Prandtl (or Schmidt) number. Thus, we are interested in the solution of Equation (12) for the case of large Prandtl number. Because of the factor $1/N_{Pr}$ multiplying the highest derivative, the equation must be transformed before solution. We follow the standard procedure of stretching the variables Y and $\hat{\theta}$ in such a way as to remove the expansion parameter(in this case the Prandtl number) from the highest order derivative. The appropriate stretching is

$$\eta = \sqrt{N_{Pr}}\,Y \tag{13}$$

and

$$F(\eta) = \sqrt{N_{Pr}}\,\hat{\theta}(Y) \tag{14}$$

Equation (12) may be rewritten in terms of F and η, and a solution of the form

$$F(\eta) = F^{(0)}(\eta) + \frac{1}{\sqrt{N_{Pr}}}F^{(1)}(\eta) + 0(N_{Pr}^{-1}) + \ldots. \tag{15}$$

is assumed and substituted into this new equation. Terms of the same order in Prandtl number are gathered, and the resulting ordinary differential equations solved. The terms of $0(N^0{}_{Pr})$ give an equation that has the solution

$$F^{(0)}(\eta) = \overline{D}e^{-(1-i)\eta} + \overline{E}e^{+(1-i)\eta} \tag{16}$$

The wall boundary condition of interest is

$$\hat{\theta}(Y=0) = 0 = F(\eta = 0) \tag{17}$$

that is, the fluctuation vanishes at the wall. This boundary condition contains two important cases, one of which is almost always found in practice, namely, those of uniform wall temperature and uniform wall heat flux. The only restriction on the latter is that

$$\frac{\partial T_{\text{wall}}}{\partial x} << \frac{\partial \overline{T}}{\partial y}$$

From Equation (17) it follows that $\overline{D} = -\overline{E}$. Then, if only the first term in Equation (15) is retained, the amplitude of the temperature fluctuation is given by

$$\hat{\theta}(Y) = \frac{F}{\sqrt{N_{Pr}}} = \frac{D}{\sqrt{N_{Pr}}}\,[e^{-(1-i)\sqrt{N_{Pr}}Y} - e^{+(1-i)\sqrt{N_{Pr}}Y}] \tag{18}$$

Since the temperature fluctuations are proportional to the mean temperature gradient, the constant $\overline{D}$ can be written $\overline{D} = D\,A/\lambda$ where A/λ is the mean temperature gradient. Equation (18) may be considered as the first term in an inner expansion for $\hat{\theta}(Y)$ for asymptotically large Prandtl number. The constant $\overline{D}$ could then be determined by matching with an appropriate outer solution. However, the inner expansion, (Equation (18)), is all that is necessary to describe the form of $\hat{\theta}$ as $Y \to 0$ and hence to determine the limiting form of the eddy diffusivity at the wall.

Equation (1), written in dimensionless form, yields

$$\frac{\epsilon_H(Y)}{\nu} = -\frac{N_{Re}}{A}\widetilde{v}\,\widetilde{\theta} \qquad (19)$$

for v and θ given in the form of Equation (3), the eddy diffusivity is described by

$$\frac{\epsilon_H(Y)}{\nu} = -\frac{N_{Re}}{2A}(\hat{v}_r\,\hat{\theta}_r + \hat{v}_i\,\hat{\theta}_i) \qquad (20)$$

where the nomenclature $N = N_r + iN_i$ is adopted for all complex numbers. Moreover, since the limit of interest is the limit as the wall is approached, Equation (18) can be examined in the limit of Y so small that $\sqrt{N_{Pr}}\,Y << 1$ even though N_{Pr} is large. Thus, Equation (18) is expanded in a power series for $\sqrt{N_{Pr}}\,Y << 1$ and the resulting equation, along with Equation (9) for $\hat{v}(Y)$, is substituted into Equation (20) to give:

$$\lim_{\sqrt{N_{Pr}}Y \to 0} \frac{\epsilon_H(Y)}{\nu} = \frac{N_{Re}}{\lambda}\{Y^3[\overline{C}_{2_r}(D_r + D_i) + \overline{C}_{2_i}(D_i - D_r)] + \frac{Y^4}{3}[A_{1_i}(D_r + D_i) - A_{1_r}(D_i - D_r)]\} \qquad (21)$$

From Sternberg's work (*12*) it is, as previously mentioned, evident that $|\overline{C}_2| \cong |A_1| \cong 1$. This suggests that the Y^3 and Y^4 coefficients in Equation (21) are of equal magnitude but does not show it conclusively. However, if the less general forms (4) and (5) are used to represent the velocity and temperature fluctuations, the ratio of these coefficients can be obtained explicitly. This will now be shown.

The procedure using (4) and (5) for v and θ is somewhat different from the foregoing analysis because now the dimensionless amplitude functions $\hat{v}(Y)$ and $\hat{\theta}(Y)$ are real. Thus, although the starting equations remain the same, each yields two equations, one from the real part and one from the imaginary part.

The phases in Equations (4) and (5) are taken to have the following linear forms near the wall:

$$\zeta = \widetilde{B}_1 Y + \widetilde{B}_2 \qquad (22)$$

and

$$\xi = \widetilde{B}_3 Y + \widetilde{B}_2 \qquad (23)$$

where $\widetilde{B}_1$, $\widetilde{B}_2$, and $\widetilde{B}_3$ are dimensionless real constants. The constant parts of ξ and ζ are assumed to be equal. This seems a good approximation for real systems of uniform wall temperature (or concentration) when the thermal capacity of the wall is much greater than the thermal capacity of the fluid. In this case the velocity and temperature fluctuations would be in phase in the limit $Y \to 0$. For finite Y the phase may differ, however, and so $\widetilde{B}_1$ may differ from $\widetilde{B}_3$. The results of the analysis for $\hat{v}(Y)$ and $\hat{\theta}(Y)$ are found to be

$$\hat{v}(Y) = a_2 Y^2 + \frac{a_1}{3} Y^3 + 0(Y^4) \qquad (24)$$

and

$$\hat{\theta}(Y) = \frac{A}{\lambda}\left(a + \frac{b}{\sqrt{N_{Pr}}}\right) Y + 0\,(N_{Pr} Y^4) \qquad (25)$$

where a, b, a_1, and a_2 are real constants.

For a given ξ and ζ the eddy diffusivity is described, from Equations (1), (4), and (5), by the equation

$$\frac{\epsilon_H(Y)}{\nu} = -\frac{N_{Re}}{2A}\hat{v}(Y)\,\hat{\theta}(Y)\cos(\zeta - \xi) \qquad (26)$$

The forms of $\hat{v}$ and $\hat{\theta}$ from Equations (24) and (25) may be substituted into Equation (26) with $\cos(\zeta - \xi)$ expanded in a power series for small Y to give the limiting form of the eddy diffusivity as

$$\lim_{\sqrt{N_{Pr}}Y \to 0} \frac{\epsilon_H(Y)}{\nu} = -\frac{aN_{Re}}{2\lambda}\left(a_2 Y^3 + \frac{a_1}{3} Y^4\right) + 0(Y^5) \qquad (27)$$

The dependence of the eddy diffusivity on Y is thus determined by the relative magnitudes of the two constants in the velocity fluctuations, a_1 and a_2. Though a_1 and a_2 are real, an estimate of them from Sternberg's work is still valid because the same equations apply in the limit as $Y \to 0$. Thus, $|a_1| \cong |a_2| \cong 1$, and hence the analysis presented here predicts that the coefficients of the Y^3 and Y^4 terms are of the same order of magnitude and that in the limit as the wall is approached the eddy diffusivity will vary as Y^3.

A word may be said regarding the relation of the results found here to experiment. For the simplified Fourier wave the analysis predicts a Y^3 dependence for the eddy diffusivity for the boundary conditions of uniform wall temperature and uniform wall heat flux. The same result is clearly suggested for a more general wave. The recent careful experiments of Harriot and Hamilton (*2*) on a system with uniform wall concentration suggest a dependence of the eddy diffusivity on Y^3 near the wall for Schmidt numbers ranging from 400 to 100,000. This conclusion has also been reached recently by several other investigators (*5* to *7*).

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NOTATION

A	=	arbitrary constant in Equation (11)
A_1	=	complex constant defined by Equation (7)
a, b	=	arbitrary constants in Equation (25), real
a_1, a_2	=	arbitrary constants in Equation (24), real
$\widetilde{B}_i$	=	real arbitrary constants in Equations (22) and (23)
C_1, C_2	=	arbitrary constants in Equation (2)
$\overline{C}_n$	=	arbitrary constants in Equation (8)
$D, \overline{D}$	=	arbitrary constants, complex
F	=	stretched variable defined by Equation (14)
K	=	constant in Equation (6), $\overline{U}/U_w = KY/\lambda$, real
k_x, k_z	=	wave numbers in the x and z directions
$\widetilde{k}_x, \widetilde{k}_z$	=	$k_x\delta_s$, $k_z\delta_s$, dimensionless wave numbers
N_{Pr}	=	Prandtl number, ν/α

N_{Re} = Reynolds number, $U_w \delta_s / \nu$
N_{Sc} = Schmidt number, ν / D
T = temperature, $T = \overline{T} + \theta$
T_o = arbitrary reference temperature
t = time
$\tilde{t}$ = $t \, U_w / \delta_s$, dimensionless time
U_i = velocity in direction i, $U_i = \overline{U_i} + u_i$
U_w = wave velocity, β / k_x
u', v', w' = root-mean square velocity fluctuations in x, y, and z directions
$\tilde{u}_k$ = u_k / U_w, dimensionless velocity fluctuation in direction k.
$\hat{v}$ = $f_2(y)/U_w$, dimensionless amplitude of the normal velocity fluctuation
x = distance parallel to the wall in the direction of the mean flow
$\tilde{x}$ = x/δ_s
y = distance normal to the wall
y^+ = $u^* y / \nu$
Y = $\sqrt{\tilde{\beta}/2\nu}\, y$

Greek Letters

β = frequency, real
$\tilde{\beta}$ = $\beta \delta_s / U_w$
δ_s = sublayer thickness
ϵ_H = eddy diffusivity defined by Equation (1)
η = stretched variable defined by Equation (13)
θ = fluctuating part of the temperature
$\tilde{\theta}$ = θ / T_o, dimensionless temperature fluctuations
$\hat{\theta}$ = dimensional amplitude of the temperature fluctuations
ν = kinematic viscosity
ζ, ξ = phases, functions of y
λ = $(N_{Re}\, \beta/2)^{1/2}$

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Concerning the Calculation of Residence Time Distribution Functions for Systems in Which Diffusion is Negligible

G. R. COKELET and F. H. SHAIR

California Institute of Technology, Pasadena, California

In order to predict the chemical performance of a reactor, a residence time distribution function must be known (*1*). The F curve is particularly convenient for such calculations. However, for reasons of economy, inlet changes in tracer concentration approximately a delta function are often used in experimental studies; for convenience the f curve is defined as the fraction of dye with residence time t or less.

When the characteristic time associated with diffusion normal to the streamlines is small compared to the characteristic residence time, the analytical method presented by Taylor (*2*) and extended by Aries (*3*), Saffman (*4*), and Lighthill (*5*) may be used. When both characteristic times are nearly equal, then a numerical solution similar to that of Farrell and Leonard (*6*) must be obtained. When the characteristic time associated with diffusion is large compared to the characteristic residence time, the method presented by Bosworth (*7, 8*) and noted by Denbigh (*9*) is applicable. However, the method of Bosworth is somewhat cumbersome and usually requires a relatively involved integration. Presented below is a calculation scheme based upon physical reasoning which requires much less work than Bosworth's method when the velocity profile can be solved explicitly for the distance coordinate. In addition, this formulation aids in understanding the relationship between the f curve and the F curve. When the velocity profile cannot be solved explicitly for the distance coordinate, both methods require about the same amount of work.

As an example of this method, we now consider the case of steady Couette flow as shown in Figure 1. Thus, we have the velocity profile

$$u = u_0 \left[1 - (y/\delta)\right] \tag{1}$$

where u_0 is the velocity at $y = 0$. The distance, x, that a fluid element travels in the time interval t is

$$x = u_0 t \left[1 - (y/\delta)\right] \tag{2}$$

In this analysis we shall consider residence time distribution in the system between the sections $x = 0$ and $x = L$.

DELTA FUNCTION

Recall the experiments reported by Goldish (*10*) and by Koutsky and Adler (*11*); that is, the fluid is consid-

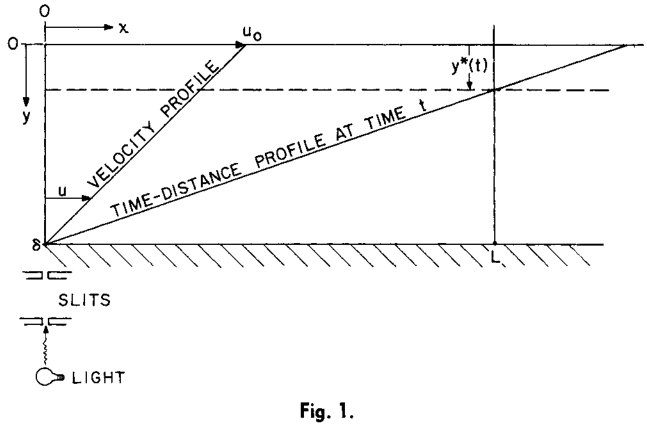

Fig. 1.

G. R. Cokelet is at Montana State University, Bozeman, Montana.